<u>WANT:</u> $C^*_{\mathrm{Lie}} : \mathrm{Lie} \longrightarrow (\mathrm{CAlg}^{\mathrm{aug}})^{\mathrm{op}}$

preserves <u>small colimits</u>

$\boxed{\text{understand these!}}$

Strategy of pf of $\uparrow$: (0) equivalent to composite

$$\begin{tikzcd}
 & (\mathrm{CAlg}^{\mathrm{aug}})^{\mathrm{op}} \arrow[dr] & \\
\mathrm{Lie} \arrow[ur, "C^*"] \arrow[dr, "C_*"'] & & (\mathrm{Mod}_k^{\mathrm{op}})_{k/} \\
 & (\mathrm{Mod}_k)_{k/} \arrow[ur, "D"'] &
\end{tikzcd}$$

preserves small colims

$V_* \longmapsto V_*^\vee$

(1) Show that C_* preserves small colims

(2) D preserves small colims b/c

$\mathrm{Vect}^{dg} \longrightarrow \mathrm{Vect}^{dg}$

$V_* \longmapsto V_*^\vee$ sends hocolims to holims

(0) uses Corollary HA 3.2.2.5: $p: \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ coCart fibr. of ∞-op., K sSet

$\forall X \in \mathcal{O}^\otimes$, $\mathcal{C}_X$ admits K-indexed limits

$\Rightarrow$ $K^\triangleleft \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is limit diag iff

$K^\triangleleft \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ is a limit $\forall X$

i.e. colims in $\mathrm{CAlg}^{\mathrm{aug}}$ are computed in $(\mathrm{Mod}_k)_{/k}$

char $k = 0$

Thm $C_* : \mathrm{Lie} \longrightarrow (\mathrm{Mod}_k)_{k/}$ preserves small colimits.

Remark • $C_* \mathfrak{g}$ has natural filtration:

$$C_* \mathfrak{g} = \mathrm{Sym}^*(\mathfrak{g}[1]) \qquad C_*^{\leq n} \mathfrak{g} = \bigoplus_{i=0}^{n} \mathrm{Sym}^i(\mathfrak{g}[1])$$

$\rightsquigarrow$ want to use this.

• want to understand "easier" colimits:

Prop It is enough to show that C_* preserves finite <u>coproducts</u> and small <u>sifted</u> colimits.

(Pf by HTT 4.2.3.11 & HA 1.3.3.10)

Recall: $K \in sSet$, $\mathcal{C} \in qCat$, $p : K \to \mathcal{C}$

Def'n: The colimit of p is any initial object of $\mathcal{C}_{p/}$.

Remark: Can identify this object w/ a map $\bar{p} : K^{\triangleright} \longrightarrow \mathcal{C}$ extending p.

Def'n: $\bar{p} : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram if it is the colim of $p = \bar{p}|_K$

$$\underset{\text{cone pt.}}{p(\infty)} := \text{"colimit of } p\text{"}.$$

~~We have seen~~ You know examples:

- $K = \bullet \amalg \bullet$ = "coproduct"
  $= \bullet \amalg \dots \amalg \bullet$ = "finite coproduct"
- $K = $
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$
  $\rightsquigarrow \mathrm{colim}_K$ = "pushout"
- $\Delta^{op} = K = $
$$\begin{tikzcd} \bullet & \bullet \arrow[l, shift left=2] \arrow[l, shift right=2] & \bullet \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3] & \cdots \end{tikzcd}$$
  colim_K = "geometric realization" $\rightsquigarrow$ generaliz.
- $K = $
$$\begin{tikzcd} \bullet & \bullet \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$
  colim_K = "split coequalizer"

  Ex: quotient: $X/G = \mathrm{colim}\left(X \leftleftarrows X \times G \right)$

  X top space, $G \curvearrowright X$

Ex:

Recall: X sSet $\quad |X| := \coprod_n X_n \times \overset{\text{geometric } n\text{-simplex}}{\Delta^n} \Big/ \sim$

$$|X| = \mathrm{colim}_{\Delta^{op}} \underbrace{X_n \times \Delta^n}_{\in \text{Space}}$$

Example: $BG = \mathrm{colim}\left(* \leftleftarrows * \times G \leftleftarrows * \times G \times G \leftleftarrows \cdots \right)$ $\quad$ ∈ Space as ∞-cat

Ex: monadic bar construction : Weibel chapt. 8

$\rightsquigarrow$ get free resolution.

If property is true for free guys & is preserved under geom. real.

$\Rightarrow$ true for all

filtered colimits:

Recall: A partially ordered set A is filtered if every finite subset of A has an upper bound in A.

Ex: • $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$ $\mathbb{Z}_{\geq 0}$-colimit is colimit of sequence

$$X_0 \to X_1 \to X_2 \to \dots \quad \text{i.e. direct limit}$$

eg. $\mathrm{Open}(X)$: $U_0 \subset U_1 \subset U_2 \subset \dots$

A filtered top. category is a top. category s.t.

(1) $\forall \{X_i\}$ finite collection $\exists X \in \mathcal{C}$ w/ $\phi_i : X_i \to X$

(2) $f, g : X \to Y$ in $\mathcal{C}$ $\Rightarrow$ $\exists h : Y \to Z$ s.t. $h \circ f = h \circ g$

(2') $X, Y \in \mathcal{C}$, $n \geq 0$, $S^n \to \mathrm{Map}_{\mathcal{C}}(X, Y)$ $\Rightarrow \exists$ $h : Y \to Z$ s.t. $S^n \to \mathrm{Map}_{\mathcal{C}}(X, Z)$ is nullhomotopic

Def'n κ regular cardinal, $\mathcal{C}$ ∞-cat.

$\mathcal{C}$ is κ-filtered if $\forall$ κ-small simpl. set K and every map $f : K \to \mathcal{C}$ $\exists$ $\bar{f} : K^{\triangleright} \to \mathcal{C}$ extending f

$\mathcal{C}$ is filtered if ω-filtered

Prop: $\mathcal{C}$ κ-filtered ∞-cat. $\Rightarrow$ $\exists$ κ-filtered partially ordered set A and a $\boxed{\text{cofinal}}$ map $N(A) \to \mathcal{C}$.

Compact objects

Def'n $\mathcal{C}$ category which admits filtered colims.

An object $C \in \mathcal{C}$ is compact if the corepresentable functor $\mathrm{Hom}_{\mathcal{C}}(C, \cdot)$ commutes w/ filtered colims

Motivating ex: X top space, $\mathcal{C} = \mathrm{Open}(X)$.

$U \in \mathcal{C}$ is compact $\iff$ U is compact as a top. space (every open cover admits a finite subcover)

Ex: $\mathcal{C} = \mathrm{Set}$. $\underset{\mathrm{Set}}{\overset{\cap}{C}}$ is cpct $\iff$ C is finite

Ex: $\mathcal{C} = \mathrm{Gp}$ $\quad G$ is cpct $\iff$ G is finitely presented

sifted colimits

cofinal maps: If $f: A \longrightarrow B$ is cofinal, then

colimits of diagrams $B \xrightarrow{p} \mathcal{C}$

are

colimits of diagrams $A \xrightarrow{f} B \xrightarrow{p} \mathcal{C}$.

$\rightsquigarrow$ allow to compute colimit over a smaller / simpler diagram!

Def'n $f: S \to T$ map of sSet. f is cofinal if

$\forall$ right fibration. $X \longrightarrow T$,

$$\mathrm{Map}_T(T, X) \xrightarrow{\simeq} \mathrm{Map}_T(S, X).$$

$$\begin{tikzcd} & X \arrow[d] \\ S \arrow[ur, dashed] \arrow[r, "p"'] & T \arrow[u, dashed, bend right] \end{tikzcd}$$

Prop: (1) Any isomorphism of simp sets is cofinal.

(2) $f: S \longrightarrow T$ cofinal $\Rightarrow$ f weak homotopy equivalence

Prop: $f: S \to T$ map of (small) simpl. sets. Equiv. are:

(1) f is cofinal

(2) $\mathcal{C}$ ∞-cat, $p: S \to \mathcal{C}$ diagram in $\mathcal{C}$, $p' = p \circ f$

$\Rightarrow$ $\mathcal{C}_{p/} \longrightarrow \mathcal{C}_{p'/}$ is equivalence of ∞-cats.

(3) $\forall$ $\mathcal{C}$ ∞-cat, $\bar{p}: S^{\triangleright} \to \mathcal{C}$ colim of $p: S \to \mathcal{C}$,

$\Rightarrow$ $\bar{p}': T^{\triangleright} \longrightarrow \mathcal{C}$ is colim of $p' = \bar{p}'|_T$.

Example: $* \hookrightarrow \mathcal{C}$ of terminal obj. $*$ is final

Example: $\mathrm{diag}: N(\Delta)^{op} \longrightarrow N(\Delta)^{op} \times N(\Delta)^{op}$

Def'n A simplicial set K is <u>sifted</u> if

(1) K is nonempty

(2) $\mathrm{diag}\, K \longrightarrow K \times K$ is cofinal

Ex: $N(\Delta)^{op}$ i.e. geometric realizations are sifted colimits.

Ex: filtered $\Rightarrow$ sifted

Prop: K sifted. Then K is weakly contractible

Pf: Choose $x \in K$ vertex. $\delta : K \to K \times K$ cofinal

$$\Rightarrow \quad \pi_n(|K|, x) \longrightarrow \pi_n(|K \times K|, x) \simeq \pi_n(|K|, x) \times \pi_n(|K|, x)$$

is bijective.

$\pi_n(|K|, x)$ non-empty $\Rightarrow$ $\simeq *$.

Motivation: (1) <u>Eilenberg-Zilber:</u>

X, Y top spaces.

$$C_\cdot(X \times Y) \xrightarrow{\simeq} C_\cdot(X) \otimes C_\cdot(Y)$$

is Corollary of:

$A : \Delta^{op} \times \Delta^{op} \longrightarrow Ab$ bisimpl. ab. gp. (obj in ab. cat.) $\rightsquigarrow$ double complex $C(A)$

$$\Rightarrow \quad C_\cdot(\mathrm{diag}(A)) \xrightarrow{\simeq} \mathrm{Tot}\, C_\cdot(A)$$

cplx assoc. to $\Delta^{op} \to \Delta^{op} \times \Delta^{op} \to Ab$

total cplx of double complex $C(A)$

In above: $C_\cdot(X \times Y) := C_\cdot(\mathbb{Z}[\underset{\text{simpl. ab gp}}{\mathrm{Sing}\, X \times \mathrm{Sing}\, Y}]) = C_\cdot(\mathbb{Z}[\mathrm{Sing}\, X] \otimes \mathbb{Z}[\mathrm{Sing}\, Y])$

$\xrightarrow{\simeq} \mathrm{Tot}\, C_\cdot(\mathbb{Z}[\mathrm{Sing}\, X]) \otimes C_\cdot(\mathbb{Z}[\mathrm{Sing}\, Y]) \simeq \mathrm{Tot}\, C_\cdot(X) \otimes C_\cdot(Y)$

② bisimplicial set $X_{\bullet,\bullet}$ (eg simplicial space)

$$|X_{\bullet,\bullet}| \simeq \operatorname{diag} X_{\bullet,\bullet} \quad \text{as sSets.}$$

as sSpace

useful prop.

~~Thm~~ **Prop**: $f: \mathcal{C} \to \mathcal{D}$ functor of ∞-cats, $\mathcal{C}$ admits small colims.

Then f preserves sifted colims iff f —"— filtered colims & geometric realizations

$f: \mathcal{C} \to \mathcal{C}'$, $\mathcal{C}$ admits κ-small colims.

HTT Prop 4.2.3.11 $\mathcal{C}$ ∞-cat, $\tau \ll \kappa$ regular cardinals.

Then $\mathcal{C}$ admits κ-small colimits iff
f preserves —"—

$\mathcal{C}$ admits τ-small colimits and colimits indexed
f preserves
by (nerves of) κ-small τ-filtered partially ordered sets

HA. Lemma 1.3.3.10 $\mathcal{C}$ ∞-cat. $f: \mathcal{C} \to \mathcal{C}'$, admit finite copr. + geom. real

If $\mathcal{C}$ admits finite coproducts & geometric realizations
f preserves "

$\Longrightarrow$ $\mathcal{C}$ admits all finite colimits
f preserves —"— "is right exact" (Prop. 5.3.2.9)

follows from

HTT Proposition 4.4.3.2 $\mathcal{C}$ ∞-cat, κ regular cardinal.

Then $\mathcal{C}$ has all κ-small colims iff

$\mathcal{C}$ has coequalizers and κ-small coproducts.

What now?

Recall: (C) Need to show: (HTT Prop 4.2.3.11 & HA Lemma 1.3.3.10)

C_* preserves finite coproducts and small sifted colimits.

Pf: (next time)

(1) preserves small sifted colimits

(a) ~~Sko~~ reduce to $\mathrm{Lie}_k \to (\mathrm{Mod}_k)_{k/} \to \mathrm{Mod}_k$ pres. sifted colims

(b) reduce to

$$C_*^{\leq n} : \mathrm{Lie}_k \to \mathrm{Mod}_k \qquad \text{—"—}$$

(c) induction over n

$$C_*^{\leq n-1} \to C_*^{\leq n} \to \mathrm{Sym}^n_k \circ \underline{\theta[1]}$$

reduces to Sym^n pres. sifted colims $\qquad \theta : \mathrm{Lie} \to \mathrm{Mod}$ forgetful pres.

(d) Sym^n is retract of

$$V_* \longmapsto V_*^{\otimes n} \quad \Rightarrow \text{pres. sifted colims}$$

(2) preserves finite coproducts

(a) C_* preserves initial objects by def'n
$\Rightarrow$ enough to show for pairwise coprods

(b) $\mathfrak{g}''_* = \mathfrak{g}_* \amalg \mathfrak{g}'_*$. need to show

$$\begin{tikzcd} k \arrow[r] \arrow[d] & C_*(\mathfrak{g}_*) \arrow[d] \\ C_*(\mathfrak{g}'_*) \arrow[r] & C_*(\mathfrak{g}''_*) \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Idea: $\mathfrak{g}_* =$ geom. realization (& sifted $\to C_*$ pres. these by (1)) of diagram of free guys $(\mathfrak{g}_\bullet)_n$